Quantum Phase Transitions and the Hidden Order in a Two-Chain Extended Boson Hubbard Model at Half-Odd-Integer Fillings

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We study the phase diagram of two weakly coupled one-dimensional dipolar boson chains at half-odd-integer fillings. We find that the system contains a rich phase diagram. Four different phases are found. They are the Mott insulators, the single-particle resonant superfluid, the paired superfluid, and the bond- or inter-chain density waves. Moreover, the Mott insulating phase can be further classified according to a hidden string order parameter, which is analogous to the one investigated recently in the one-dimensional boson Mott insulator at integer fillings.

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I. INTRODUCTION

The low dimensional electron and spin systems are usually regarded as important playgrounds for the studies of correlated quantum matters due to the strong quantum fluctuation effects in one and two spatial dimensions. Many exotic non-Fermi liquid states such as the Luttinger liquid and spin liquid states, to name just a few, are found in the single or coupled chain systems. However, nature never stops surprising us. The recent advances in the technique of loading ultracold gases into optical lattices open a new era in the research of condensed matter systems. Among the recent achievements, the experimental realization of the Boson-Hubbard model signatures an important step in this direction, not only because of the tunability of the controlling parameters in the corresponding experiments, but also because it facilitates the first observation of the Mott insulating state of bosons and the associated quantum phase transition.[1] While the atomic interactions in the ultracold gases can be treated as contact ones for most cases, a sizable longer range interaction is now within experimental reach by using the dipolar interaction among atoms,[2,3,4] which could provide further opportunities of controlling/designing new experiments. Following these lines of developments, a natural question to ask is whether or not the longer range interactions can trigger a stable new quantum phase of matter which contains non-trivial internal structure.

Among many research works devoted to the understanding of the effects of long-range dipolar interactions, we mention the recent work by Emanuele G. Dalla *et al*,[5] who studied the one-dimensional boson insulators within the context of an extended boson Hubbard model (EBHM) by employing the density matrix renormalization group (DMRG) method. By tuning the ratio of the on-site interaction over the hopping amplitude U/t and the ratio of the longer range interaction over the hopping amplitude V/t, the mean field analysis shows that three different *conventional* phases can be reached. These include the Mott insulator at large U, the density wave state for large V and a superfluid state for large t. The surprising thing is that a new intermediate insulating state, the Haldane insulator, which separates itself from the other two insulating states by second order quantum phase transitions was found in Ref. 5. Moreover, it was shown that such a state possesses a non-vanishing nonlocal string order, similar to the Haldane phase of the quantum spin-one chain. Such a state is definitely beyond the reach of the traditional one-dimensional hydrodynamic effective theory for the one-dimensional boson superfluid-to-Mott transition.[8] Recently, the present author and his collaborators have developed an phenomenological *two-component* hydrodynamical effective theory[9] which successfully captures the main features of all the phases, including the Haldane insulator, found in the recent DMRG study of the one-dimensional EBHM at integer fillings. This effective theory also clarifies the nature of the quantum phase transitions between different phases.

Knowing the above results, it is desirable to see if similar exotic phases or insulating states can be found in other one-dimensional or quasi-one-dimensional systems. As a first attempt, we consider in this paper a dipolar boson system of two weakly coupled chains within the framework of the EBHM. Interestingly, we found that the competition of the inter-chain hopping and the inter-chain interaction does lead to two different types of Mott insulating states, with one of them possessing a nontrivial string order. In addition to that, we also found that the inter-chain attraction can give rise to an interesting *paired* superfluid state where the inter-chain bound boson pairs show an algebraic long range superfluid order while the single-boson superfluid correlations decay exponentially. The rest of the paper is organized as follows: In section II, we introduce our model and its effective theory. In section III and IV, we analyze the phases of the model and discuss the issue of the string order in the Mott insulating state. The final section is dedicated to our conclusions, and the resulting phase diagram is summarized in figure 2.

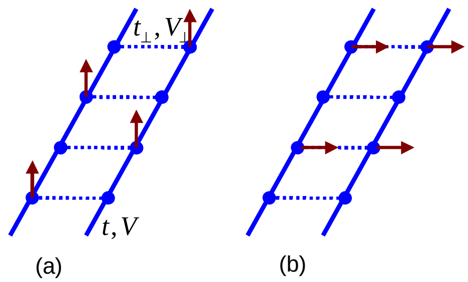

FIG. 1: (color online) The two-chain lattice model described by Hamiltonian (1). The arrows denote the possible orientations of the dipole moments. The left figure (a) corresponds to the IDW phase and the right figure (b) represents the BDW phase considered in the paper. (See section III.)

II. THE MODEL AND ITS EFFECTIVE THEORY

The lattice bosons studied in the present paper is described by the following extended Bose-Hubbard model

$$
\begin{aligned}
H = & -t \sum_{\sigma,\langle i,j\rangle} (b^\dagger_{\sigma i} b_{\sigma j} + \text{H.c.}) + \frac{U}{2}\sum_{\sigma i} \delta n^2_{\sigma i} \\
& + V \sum_{\sigma,\langle i,j\rangle} \delta n_{\sigma i}\, \delta n_{\sigma j} \\
& + t_\perp \sum_i (b^\dagger_{1i} b_{2i} + h.c.) + V_\perp \sum_i \delta n_{1i} \delta n_{2i}\,, \qquad (1)
\end{aligned}
$$

where $\sigma = 1, 2$ is the chain index, t is the nearest-neighbor hopping along the chain, $t_\perp$ is the inter-chain hopping, U is the on-site repulsion, and $V, V_\perp$ are the nearest-neighbor interactions along and between the chains, respectively. The operator $b^\dagger_{\sigma i}$ create a boson at site i on chain σ, $n_{\sigma i} = b^\dagger_{\sigma i} b_{\sigma i}$ is the number operator, and $\delta n_{\sigma i} = n_{\sigma i} - \bar{n}$ measures the deviation of the particle number from a mean filling $\bar{n}$. In the present paper, we will focus our attention to the cases of half-odd-integer fillings, i.e. $\bar{n} = N + 1/2$ where N is a non-negative integer.

We stress that the above model can be realized by polar molecules, [2,3] or atoms with a larger dipolar magnetic moment such as ^{53}Cr.[4] Moreover, it is possible to adjust the mutual orientations between the dipoles by using external electric (magnetic) fields, so that we may control the signs of the nearest-neighbor interactions V and $V_\perp$. Therefore, both the cases of the attractive and the repulsive nearest-neighbor couplings will be considered below. (See Fig. 1.) With this understanding in mind, we now discuss the low energy effective theory of the model defined in Eq. (1).

For simplicity, we first consider the case where the on-site repulsion U is the largest energy scale, i.e. $U \gg t, V, t_\perp, V_\perp$. Under this condition and near half-odd-integer fillings, we can truncate the boson Hilbert space so that only states with local occupations $n_{\sigma i} = N+1$ and N are allowed, and the boson operators can be mapped to spin-1/2 operators, $b^\dagger_{\sigma i} \to S^+_{\sigma i}$, $\delta n_{\sigma i} \to S^z_{\sigma i}$. In this low-energy subspace, the extended boson Hubbard model is mapped to coupled xxz spin-1/2 models,

$$
\begin{aligned}
H &= H_0 + H_\perp \\
H_0 &= -t \sum_{\langle i,j\rangle} (S^+_{1i} S^-_{1j} + h.c.) + V \sum_{\langle i,j\rangle} S^z_{1i} S^z_{1j} + (1 \to 2) \\
H_\perp &= t_\perp \sum_i (S^+_{1i} S^-_{2i} + h.c.) + V_\perp \sum_i S^z_{1i} S^z_{2i}\,. \qquad (2)
\end{aligned}
$$

In the following, we will assume that the inter-chain coupling $t_\perp, V_\perp$ are much smaller than the intra-chain couplings t and V. Then the spin-1/2 chain can be bosonized using the standard method,[7] and the intra-chain Hamiltonian can be written in the following form,

$$
\begin{aligned}
H_0 = & \sum_\sigma \frac{v}{2} \int dx\ \frac{1}{K} (\partial_x \phi_\sigma)^2 + K (\partial_x \theta_\sigma)^2 \\
& + g \int dx\ \cos\sqrt{4\pi}\phi_\sigma\,. \qquad (3)
\end{aligned}
$$

At the perturbative level, $t \gg V$, we have $v = v_0 \sqrt{1 + \frac{2V}{\pi t}} \equiv \frac{v_0}{K}, v_0 = t a_0$, and $g = \frac{V}{4\pi^2 a_0}$ (a_0 is the lattice spacing.). However, the validity of this effective action goes beyond its perturbative derivation, and the relations between the Luttinger parameter K and v and the spin-chain couplings t, V can be established exactly, [6]

$$
v = v_0 \frac{\pi\sqrt{1-\Delta^2}}{2\cos^{-1}\Delta}, \quad K = \frac{\pi}{2(\pi - \cos^{-1}\Delta)}\,, \qquad (4)
$$

with $\Delta = \frac{V}{2t}$ in the above equation.

The inclusion of the inter-chain coupling terms do not present any new difficulty. Upon including them, the resulting Hamiltonian becomes

$$
\begin{aligned}
H = & \sum_{\alpha=s,a} \frac{u_\alpha}{2} \int dx\ \frac{1}{K_\alpha} (\partial \phi_s)^2 + K_\alpha (\partial \theta_s)^2 \\
& + 2g \int dx \cos\sqrt{4\pi}\phi_s \cos\sqrt{4\pi}\phi_a + \frac{t_\perp}{\pi a} \int dx\ \cos\sqrt{2\pi}\theta_a \\
& + \frac{V_\perp}{2\pi a_0} \int dx\ \left(\cos\sqrt{8\pi}\phi_a - \cos\sqrt{8\pi}\phi_s\right). \qquad (5)
\end{aligned}
$$

In the above equation, ϕ_s, θ_s and ϕ_a, θ_a are the symmetric and anti-symmetric combinations of the boson fields for the spin-1/2 operators, respectively. In terms of them, the original lattice boson operators can be expressed as,

$$
\frac{b_{1/2,i}}{\sqrt{a_0}} = \frac{1}{\sqrt{2\pi a_0}} e^{i\sqrt{\frac{\pi}{2}}(\theta_s \pm \theta_a)} \left(1 + (-1)^{x/a} \sin\sqrt{2\pi}(\phi_a \pm \phi_a)\right) \qquad (6)
$$

and the bond and inter-chain density fluctuations are

$$
\frac{\delta n_{s/a}}{a_0} = \sqrt{\frac{2}{\pi}} \partial_x \phi_{s/a} + (-1)^x \frac{-2}{\pi a_0} \begin{cases} \sin\sqrt{2\pi}\phi_s \cos\sqrt{2\pi}\phi_a \\ \cos\sqrt{2\pi}\phi_s \sin\sqrt{2\pi}\phi_a \end{cases} . \qquad (7)
$$

Since the intra-chain nonlinear term $g\cos\sqrt{8\pi}\phi_s\cos\sqrt{8\pi}\phi_a$ is always less relevant than the inter-chain coupling terms in the region that we are interested in below, it can be safely neglected. Therefore, in the strong on-site repulsion limit, the two-chain extended boson Hubbard model at half-odd-integer fillings can be described by the following effective hydrodynamic theory,

$$\begin{aligned}H = &\sum_{\alpha=s,a}\frac{u_\alpha}{2}\int dx\ \frac{1}{K_\alpha}(\partial\phi_s)^2+K_\alpha(\partial\theta_s)^2\\&+g_1\int dx\ \cos\sqrt{2\pi}\theta_a\\&+g_2\int dx\ \cos\sqrt{8\pi}\phi_a+g_3\int dx\ \cos\sqrt{8\pi}\phi_s\,,\end{aligned}\tag{8}$$

where $g_1=\frac{t_\perp}{\pi a_0}, g_2=-g_3=\frac{V_\perp}{2\pi a_0}$, $v_{s,a}=v\sqrt{1\pm K\frac{V_\perp a_0}{\pi v}}, K_{s,a}=\frac{K}{\sqrt{1\pm K\frac{V_\perp}{\pi v}}}$, and v,K here are parameters defined in Eq. (4). At this point, it is interesting to notice that exactly an action of the same form occurs in our recent study of the one-dimensional boson Mott transition near integer fillings,[9] albeit in a completely different physical context.

Before we analyze the consequences of the Hamiltonian in Eq. (8), we emphasize that this effective action is quite general and is not restricted to hard-core bosons. For the finite U soft-core bosons, one may introduce the boson field and density operators,[8]

$$\begin{aligned}\frac{b^\dagger_{i\sigma}}{\sqrt{a_0}}&=(\rho_0+\frac{1}{\sqrt{\pi}}\partial_x\phi_\sigma)^{1/2}\sum_p e^{i2p(\pi\rho_0x+\sqrt{\pi}\phi_\sigma)}e^{i\sqrt{\pi}\theta_\sigma}\\\frac{n_{i\sigma}}{a_0}&=\left(\rho_0+\frac{1}{\sqrt{\pi}}\partial_x\phi_\sigma\right)\sum_p e^{i2p(\pi\rho_0x+\sqrt{\pi}\phi_\sigma)}\,,\end{aligned}\tag{9}$$

where $\sigma=1,2$ are the chain indices, and $\rho_0\approx(N+\frac{1}{2})/a_0$ is the average boson density. After some simple manipulations, it is not hard to see that we will still get the same effective action (8). Consequently, the results we get in this paper is robust as long as the inter-chain couplings are weak. The only problem is that in soft-core boson case, the relations between the Luttinger parameters $K_{s,a}$ and the microscopic couplings are unknown and Eq. (8) can only be treated in a phenomenological manner.

III. THE PHASES AND PHASE TRANSITIONS OF THE EXTENDED BOSON HUBBARD MODEL

After establishing our effective theory for the two-chain EBHM, we now turn to study the possible phases and the associated quantum phase transitions of this system. The first thing to notice is that the charge U(1)×U(1) symmetry of the decoupled chains is broken down to a U(1)×Z_2 by the inter-chain hopping term $t_\perp(b^\dagger_{1i}b_{2i}+h.c.)$, which can be viewed as the inter-chain Josephson coupling between two one-dimensional boson superfluids. We also notice that the diagonal U(1) symmetry here forbids terms like $\cos\beta\theta_s$. In addition to that, there is a lattice translation symmetry of the EBHM. As we shall see immediately, these discrete symmetries can be broken spontaneously due to correlation effects.

The only nonlinear term which can potentially open a gap in the symmetric mode is the g_3 term in Eq. (8), and its dimension is $2K_s$. Therefore, we expect that depending on whether $K_s>1$ or $K_s<1$, the symmetric hydrodynamic mode will either be gapless or acquire a gap. Furthermore, the transition corresponding to the opening of the spectral gap in the symmetric mode falls into the KT universality class. On the other hand, for the antisymmetric mode, there are two competing nonlinear terms g_1 and g_2 in Eq. (8), which have scaling dimensions $\frac{1}{2K_a}$ and $2K_a$, respectively. Hence, depending on whether $K_a>1/2$ or $K_a<1/2$, either the $g_1\cos\sqrt{2\pi}\theta_a$ term or the $g_2\cos\sqrt{8\pi}\phi_a$ term will be dominant and opens a gap in the anti-symmetric mode. In other words, the anti-symmetric mode will always be gapped except at the critical line defined by $K_a=1/2$, where the critical theory falls into the Ising universality class. The nature of this transition and the associated critical mode can be seen as follows:[7] Exactly at $K_a=1/2$,[10] we may introduce a set of right- and left-moving Dirac fermions $\chi_{R,L}=\frac{1}{\sqrt{2\pi a}}e^{\pm\sqrt{4\pi}\phi'_{R,L}}$ with $\phi'_{R,L}=\frac{1}{\sqrt{2K_a}}\Phi_a+\mp\sqrt{\frac{K_a}{2}}\Theta_a=\Phi_a+\mp\sqrt{\frac{1}{4}}\Theta_a$ to rewrite the Hamiltonian. In order to identify the critical mode and the physical low lying excitations, it is convenient to further decompose the Dirac fermions into their real and imaginary parts $\chi_\nu=\frac{1}{\sqrt{2}}(\xi_\nu+i\rho_\nu), \nu=R,L$, and express the Hamiltonian in terms of these Majorana fermions

$$\begin{aligned}H_a=&\frac{-iu_a}{2}(\xi_R\partial_x\xi_R-\xi_L\partial_x\xi_L)+im_\xi\xi_R\xi_L\,,\\&+\frac{-iu_a}{2}(\rho_R\partial_x\rho_R-\rho_L\partial_x\rho_L)+im_\rho\rho_R\rho_L\,.\end{aligned}\tag{10}$$

In the above equation, the Majorana fermion masses are $m_\xi=\frac{g_2+g_3}{4\pi a}$, $m_\rho=\frac{g_2-g_3}{4\pi a}$. When $|g_2|=|g_3|$, one of the Majorana fermion will become gapless and the remaining one will still be massive. Hence, we see that the competition between the inter-chain tunnelling and the inter-chain interaction may result in an Ising transition. The above discussions suggest that, depending on $K_s\lessgtr 1$ and $K_a\lessgtr 1/2$, there can be four different phases. In the following, we shall elaborate on the natures of these four phases.

We start by considering the case where $K_s>1$ and $K_a>1/2$. In this case, the only relevant part of the interaction Hamiltonian is the g_1-term, which arises from the inter-chain Josephson coupling. Using Eq. (6) and Eq. (7), one may easily see that the single particle and

density correlations are

$$\langle b_{1i}^\dagger b_{1j}\rangle = \langle b_{2i}^\dagger b_{2j}\rangle \sim \frac{1}{|i-j|^{1/4K_s}} \ ,$$
$$\langle \delta n_{\sigma i}\delta n_{\sigma j}\rangle \sim \frac{1}{|i-j|^2} + \begin{pmatrix}\text{an exponentially decaying}\\ \text{staggerred part}\end{pmatrix} \ . \tag{11}$$

It is also interesting to examine the following long distance two-particle correlations,

$$\langle b_{1i}^\dagger b_{2i}^\dagger b_{1j} b_{2j}\rangle \sim \frac{1}{|i-j|^{1/K_s}} \,, \tag{12}$$
$$\langle b_{1i}^\dagger b_{2i} b_{1j}^\dagger b_{2j}\rangle \sim \ \text{const.} \tag{13}$$

From the above equations, we see that the single-particle correlation is dominant over the two-particle pair correlation, as is expected for a usual superfluid. However, Eq. (13) also shows that the system exhibits a true long-range order for the pair condensate $\langle b_{1i}^\dagger b_{2i}\rangle \neq 0$. The existence of a non-zero condensate in this one-dimensional system indicates that some kind of resonant bond-pairs are formed for bosons on two different chains, and it is a direct manifestation of the phase-locking between the two chains due to the inter-chain Josephson coupling. We also notice that although the density correlations along the chain decay algebraically, the density-difference $\delta n_{-i} =: b_{1i}^\dagger b_{1i} - b_{2i}^\dagger b_{2i}$: has an exponentially decaying correlation function. This is consistent with the above resonant bonding-boson pair picture. Since there is no translational symmetry breaking in this phase, and motivated by Eq.(11) and Eq.(13), this phase will be coined as the single-particle resonant superfluid (RSF) later in this paper.

We next consider the second gapless phase specified by the Luttinger parameters $K_s > 1$ and $K_a < 1/2$. In this case, the relevant part of the interacting Hamiltonian is g_2-term which originates from the inter-chain dipolar interaction between bosons. The single particle and density correlations in this phase are

$$\langle b_{1i}^\dagger b_{1j}\rangle = \langle b_{2i}^\dagger b_{2j}\rangle \sim \ \text{decays exponentially} \ \ ,$$
$$\langle \delta n_{\sigma i}\delta n_{\sigma j}\rangle \sim \frac{1}{|i-j|^2} + (const.)\times\frac{(-1)^{|i-j|}}{|i-j|^{K_s}} \,. \tag{14}$$

The most interesting feature of this phase is that although the symmetric mode is gapless and possess a non-zero superfluid stiffness, the single-particle superfluid correlation decays exponentially. On the other hand, two-particle correlation shows an algebraic long range order,

$$\langle b_{1i}^\dagger b_{2i}^\dagger b_{1j} b_{2j}\rangle \sim \frac{1}{|i-j|^{1/K_s}} \,,$$
$$\langle b_{1i}^\dagger b_{2i} b_{1j}^\dagger b_{2j}\rangle \sim \ \ \text{decays exponentially} \ . \tag{15}$$

From the above results, we conclude that this translationally invariant phase is a one-dimensional *paired superfluid* (PSF), where the inter-chain bond-boson pairs flow coherently along the chain.

We now turn to discuss the remaining two gapped phases. For $K_s < 1$ and $K_a > 1/2$, both the inter-chain Josephson and the inter-chain dipolar interaction are relevant, and these interactions open gapes for both the symmetric and ant-symmetric modes. The relevant part of the effective Hamiltonian in the present case is,

$$H = \sum_{\alpha=s,a}\frac{v_\alpha}{2}\int dx\left[\frac{1}{K_\alpha}(\partial_x\phi_\alpha)^2 + K_s(\partial_x\theta_\alpha)^2\right] + g_1\int dx\ \cos\sqrt{2\pi}\theta_a + g_3\int dx\ \cos\sqrt{8\pi}\phi_s\,. \tag{16}$$

In this phase, it is easy to see that among the correlation functions examined previously, the only one which does not decay exponentially is the resonant-pair condensate Eq. (13). This is a translationally invariant insulating state with a non-zero resonant pair boson condensate. Therefore, it is a Mott insulator (MI).

The last phase corresponds to $K_s < 1, K_a < 1/2$, and the relevant part of the effective Hamiltonian in this case is,

$$H = \sum_{\alpha=s,a}\frac{v_\alpha}{2}\int dx\left[\frac{1}{K_\alpha}(\partial_x\phi_\alpha)^2 + K_s(\partial_x\theta_\alpha)^2\right] + g_2\int dx\ \cos\sqrt{8\pi}\phi_a + g_3\int dx\ \cos\sqrt{8\pi}\phi_s\,, \tag{17}$$

with $-g_3 = g_2 \propto V_\perp$ in the simplest EBHM. In this phase, all the single-particle and the pair correlations decay exponentially. On the other hand, for the bond- and inter-chain density fluctuations, $\delta n_{\pm i} \propto: b_{1i}^\dagger b_{1i} \pm b_{2i}^\dagger b_{2i}$:, we have

$$\begin{cases} V_\perp > 0, & \langle\delta n_{-i}\rangle \sim (-1)^{x_i}/a_0, \ , \\ V_\perp < 0, & \langle\delta n_{+i}\rangle \sim (-1)^{x_i}/a_0 \end{cases} \ . \tag{18}$$

This result indicates that the discrete Z_2 lattice translation symmetry is broken spontaneously. Therefore, depending on the sign of the inter-chain dipolar interaction, the system exhibits either a bond-density wave pattern (BDW) or an inter-chain density wave pattern (IDW). (See Fig.1 for a pictorial illustration.) Notice that if we change the relative orientations of the dipoles between the two chains so that the inter-chain dipolar interaction changes its sign, we will tune a quantum phase transition across these two different density wave phases and the associated critical theory is a U(1)×U(1) Gaussian theory. These translationally non-invariant states can, in principle, be observed experimentally in the corresponding spatial image patterns of the time-of-flight experiments.

IV. THE HIDDEN STRING ORDER

Inside the Mott insulator phase, we can also adjust the relative orientations of the dipoles between the two chains to change the sign of $V_\perp$. This will introduce a further quantum phase transition due to the accidental

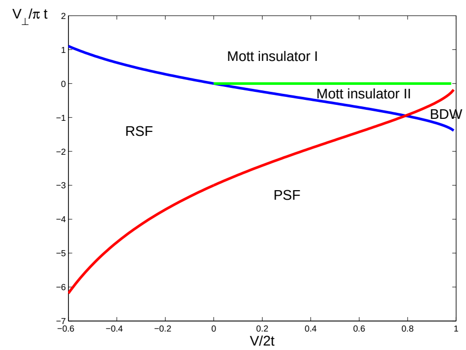

FIG. 2: (color online) The phase diagram of the two weakly coupled half-filled one-dimensional EBHM obtained in the hard-core boson limit. Notice that for the hard-core bosons, only the BDW phase can occur. However, we expect that this is a artifact of the hard-core condition, and for more general cases, depending on the sign of $V_\perp$, both the IDW and the BDW phases can be stablized. See the discussions in the last section. Notice also that the phase transition across the red line is of the Ising universality class, while the phase transition across the blue line belongs to the KT universality class. The Mott insulator II below the green line contains a non-trivial string order, and it is separated from the Mott insulating I state by a U(1) Gaussian critical theory.

gap closing in the symmetric mode alone, and the corresponding critical theory is a U(1) Gaussian theory. On both sides of the transition, the system is gapped and do not break any lattice symmetry. So, what is the difference between these two Mott insulating states? Interestingly, from the structure of the above hydrodynamic theory and the results of recent studies on the 1D boson Mott insulator, [5,9] we expect that one of the Mott insulating phase will posses a non-vanishing non-local string order, [11] while the other does not. This can also be seen by noticing that our effective Hamiltonian bears some similarities with that of *isotropic* spin ladders,[12,13] where nontrivial string orders were also found. More specifically, drawing the analogy with the isotropic spin ladders, we may introduce the string operator

$$\hat{O}_{string}(i-j) = \lim_{|x-y|\to\infty} \langle \delta n_{+i} e^{i\pi \sum_{l=i+1}^{j-1} \delta n_{+l}} \delta n_{+j} \rangle \,. \quad (19)$$

Following of the discussions of M. Nakamura,[14] we expect that the correct bosonized form of the string operator should be

$$\hat{O}_{string} \sim \lim_{|x-y|\to\infty} \left\langle \sin\sqrt{2\pi}\phi_s(x) \sin\sqrt{2\pi}\phi_s(y) \right\rangle \,, \quad (20)$$

which is nonzero for $V_\perp < 0$, due to the pining of the symmetric mode at $\phi_s = \sqrt{\frac{\pi}{8}}$. On the other hand, the string order vanishes for $V_\perp > 0$ where the field ϕ_s is pinned at $\phi_s = 0$. Therefore, for the negative $V_\perp$ case, the MI phase we found here is the boson analogy of the AKLT phase of a spin ladder system, while for the positive $V_\perp$ case, the boson MI in the present EBHM is the boson analogy of the RVB phase of the spin ladder.[14]

A more interesting question is perhaps about how to detect the differences between the two Mott insulating states. From the structure of the effective theory, the bulk spectrums of these two insulating states in the immediate neighborhoods of the transition line are almost identical. On the other hand, it can be shown that the MI state with a non-vanishing string order contains a zero energy edge state.[16] This fact is most easily seen by examining the Sine-Gordon model of the symmetric mode near its Luther-Emery point[10] ($K_s = 1/2$). Near the Luther-Emergy point, the symmetric mode can be mapped to a massive Dirac fermion theory,[17]

$$H = \int_0^\infty dx \Psi^\dagger(x) \left[-iv\sigma^3\partial_x + \sigma^2 m\right] \Psi(x) + \cdots \,, \quad (21)$$

where $\Psi(x) = (\psi_R(x), \psi_L(x))^T$ is a Dirac fermion field, $m = -V_\perp/2$, and $\cdots$ denotes the possible four fermion interactions. These fermions should be identified as the low energy elementary (soliton) excitations of the symmetric mode in the insulating states. For a system with a boundary at, say $x = 0$, we may introduce the boundary condition $\psi_R(0) = \psi_L(0)$,[18] and reformulate the above Dirac theory in terms of a single chiral component by introducing $\psi_R(x) = \psi_L(-x)$ for $x < 0$. After doing so, the Dirac Hamiltonian reduces to

$$H = \int_{-\infty}^\infty dx - iv\psi_R(x)^\dagger \partial_x \psi_R(x) + im\text{sgn}(x)\psi_R^\dagger(x)\psi_R(-x) \,. \quad (22)$$

It is not hard to see that the corresponding Schrödinger equation

$$-iv\partial_x\psi_R(x) + im\text{sgn}(x)\psi_R(-x) = \epsilon\psi_R(x) \,, \quad (23)$$

contains a zero energy state localized at the edge,

$$\psi_{Re}(x) \sim e^{-m|x|} \,, \; \epsilon_e = 0 \,. \quad (24)$$

Notice that this edge state exists only for $m > 0$, i.e. for $V_\perp < 0$. Since all other bulk excitations have finite gapes, the existence of this zero-energy edge state constitutes a unique feature of the string-ordered MI state, and its existence can be revealed in a zero frequency delta-function like peak in the optical absorption spectrum. Since in real experiments, the system is trapped in a finite domain, it is interesting to see if such a delta-function absorption peak can be observed experimentally.

V. CONCLUSIONS AND DISCUSSIONS

Although most of our results are obtained in the weak inter-chain coupling limit, we point out here that some of the important features of the weak coupling phase diagrams can be realized even in the strong inter-chain coupling limit. For example, the effect of a large $V_\perp$ can be understood by considering the hard-core boson limit and keeping only the states with local occupations $n_{\sigma i} = 0, 1$.

Under this condition, we may introduce the bond operators,[19]

$$
\begin{aligned}
s^\dagger|0\rangle &= \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)\,, \quad t_x^\dagger|0\rangle = \frac{-1}{\sqrt{2}}(|11\rangle - |00\rangle)\,,\\
t_y^\dagger|0\rangle &= \frac{i}{\sqrt{2}}(|11\rangle + |00\rangle)\,, \quad t_z^\dagger|0\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)\,,
\end{aligned} \tag{25}
$$

for each site, and the ket, say $|10\rangle$, denotes a state with a local bond-boson occupation $|n_{1i}=1, n_{2i}=0\rangle$, and these operators obey the constraint $s_i^\dagger s_i + \sum_\alpha t_{\alpha i}^\dagger t_{\alpha i} = 1$. In terms of these bond operators, the inter-chain interaction can be written as

$$
H_{V_\perp} = V_\perp \sum_i (t_{xi}^\dagger t_{xi} + t_{yi}^\dagger t_{yi}) - (s_i^\dagger s_i + t_{zi}^\dagger t_{zi})\,. \tag{26}
$$

When $|V_\perp| \gg |t|, |t_\perp|$, and $V_\perp > 0$, we may keep only s_i, t_{zi}, and the leading order effective low energy Hamltonian becomes

$$
\begin{aligned}
H = -V_\perp &+ \frac{V}{2}\sum_i (n_{Ai}n_{Ai+1} + n_{Bi}n_{Bi+1})\\
&- \frac{V}{2}\sum_i (n_{Ai}n_{Bi+1} + n_{Bi}n_{Ai+1}) + O(\frac{t^2, t_\perp^2}{V_\perp})\,,
\end{aligned} \tag{27}
$$

where $A_i^\dagger = \frac{1}{\sqrt{2}}(s_i^\dagger + t_{zi}^\dagger), B_i^\dagger = \frac{1}{\sqrt{2}}(s_i^\dagger - t_{zi}^\dagger)$ create bond-pairs $|10\rangle_i$ and $|01\rangle_i$, respectively, and $n_{A,Bi} = A_i^\dagger A_i, B_i^\dagger B_i$ are the number operators of the A and B bosons, respectively. We see from this strong coupling analysis that in the large $V_\perp$ limit, the ground state configuration is completely determined by minimizing the inter-particle interactions between the A and B particles. Therefore, for $V > 0$, we expect a Wigner crystal like state with the A and B particles are arranged in an alternating pattern $\cdots ABAB \cdots$, while for $V < 0$, phase separation will occur. In terms of the original two-chain boson language, this Wigner crystal like density wave state is exactly the IDW phase discussed previously. When $V_\perp < 0$, we get similar results, except now that the A and B operators are defined by $A_i^\dagger = \frac{1}{\sqrt{2}}(t_{xi}^\dagger - it_{yi}^\dagger), B_i^\dagger = \frac{1}{\sqrt{2}}(t_{xi}^\dagger + it_{yi}^\dagger)$, which create states with local-bond occupations $|11\rangle_i$ and $|00\rangle_i$, respectively. Again, for $V > 0$ the ground state is a density wave state and for $V < 0$ we have phase separations. It is not hard to see that the Wigner crystal like state in this situation is just the BDW phase discussed earlier.

Similarly, when the inter-chain hopping amplitude $t_\perp$ is the largest energy scale, we may introduce the inter-chain bonding and anti-bonding bosons fields $b_{\pm,i} = \frac{1}{\sqrt{2}}(b_{1i} \pm b_{2i})$. At low energy, the anti-bonding bosons can be projected out and the resulting low energy Hamiltonian becomes

$$
\begin{aligned}
H_{eff} = -t_\perp &\sum_i b_{+i}^\dagger b_{+i} - t\sum_i (b_{+i}^\dagger b_{+,i+1} + h.c.)\\
&+ \frac{U}{4}\sum_i (:n_{b_+i}:)^2 + \frac{V}{2}\sum_{\langle i,j\rangle} :n_{b_+i}::n_{b_+j}:\\
&+ O(\frac{U^2, V_\perp^2, V_\perp^2}{t_\perp})\,,
\end{aligned} \tag{28}
$$

to the leading order, where $n_{b_+i} = b_{+i}^\dagger b_{+i}$ is the density operator of the bonding bosons. This effective Hamiltonian is nothing but that of the one-dimensional EBHM of the bonding-bosons at integer fillings. Transforming back to the original two-chain boson language, it is not hard to see that the superfluid phase of the bonding boson at large t, where the boson density distributed uniformly, automatically possesses a non-vanishing resonant-pair condensate $\langle b_{1i}^\dagger b_{2i} + h.c.\rangle$. Hence this phase may be identified as the RSF phase of the two-chain EBHM considered in this paper. Similarly, its density wave phase at large V can be viewed as the BDW phase of the original two-chain system. According to Ref. 5, when V increases to the order of U, we expect the bonding boson will enter into a "Haldane insulator" phase which has a non-trivial string order. Since in the truncated Hilbert space where the anti-bonding bosons are projected out, the density operator n_{b_+i} coincides with the total bond boson number operator n_{+i}, Hence, the string operator defined in Eq. (19) reduces to that of Ref. 5 in the large $t_\perp$ limit. Although the transition between the Haldane insulating phase and the conventional Mott insulating phase of the bonding-bosons is achieved by tuning the intra-chain coupling V, while it is the inter-chain coupling $V_\perp$ which plays a decisive role in the corresponding phase transition of the weakly coupled chain case, the fact that the string operators coincide in certain limit suggests that these two phases are, in fact, adiabatically connected.

We now briefly discuss some earlier results which are related to the present work. A phase analogous to our PSF state was noticed in the context of a coupled spin model,[20] and it was also shown to appear in a boson ladder model at incommensurate fillings.[21] The Mott-superfluid transition for a boson ladder model, coupled by the inter-chain josephson coupling alone, at the commensurate filling of one boson per site was studied in Ref. 22, while in our present case, the Mott insulating state occurs at half-odd-integer fillings. We also notice that the hidden order in single chain electronic Hubbard models was studied using both the bosonization and the DMRG methods in Ref. 23, and there are some recent efforts which address the issue of string orders in electronic band insulators and their relations with the Haldane chain.[24] These works, together with the results of Ref. 5 and our work, suggest that the idea of classifying the boson or fermion Mott insulators in quasi-one-dimensional systems using the string order may be guite general.

To summarize, the above strong coupling analysis shows that the phases we obtained, when one of the inter-chain coupling is dominant, is consistent with the results we gained in the weak-coupling bosonization analysis. Since one of the most important conclusion that follows from the bosonization study is that the interplay between the inter-chain hopping and inter-chain interaction results in two exotic phases — the paired superfluid state and the Mott insulating state with a string order, it is tempting to speculate that the major results of the bosonization analysis in this paper should be valid even when the inter-chain coupling strength is not so weak and can be extended to larger regions in the whole phase diagram.

Using the analytic results obtained in this paper as a guide, we hope that it is helpful for future numerical works to determine the exact phase boundaries of the two-chain EBHM, and more importantly, the exotic insulating phase and the paired superfluid phase can be observed in the future experiments.

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